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MODELS IN GAME THEORY

by

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I. Introduction

1. The nature of game theory. Game theory is a collection of mathematical models formulated to study decision making in situations involving conflict and cooperation. It recognizes that conflict arises naturally when various participants have different preferences, and that such problems can be studied quantitatively. Game theory attempts to abstract out those ingredients which are common and essential to many different competitive situations and to study them by means of the scientific approach. It is concerned with finding optimal solutions, stable outcomes, or equitable allocations when various decision makers have conflicting objectives in mind. These models describe how one should proceed in order to arrive at the best possible outcome in light of the options open to one's opponents. Game theory thus attempts to provide a normative guide to rational behavior when acting in a group whose members aim for different goals. In general a game consists of players who must choose from a list of alternatives which will in turn bring about certain outcomes over which the participants may have different preferences. Chance and random events may also influence the final payoffs.

2. Related subjects. Game theory is concerned with two or more participants involved in strategic encounters (games of skill). It thus extends beyond the classical theories of probability (games of chance) and decision theory (games against nature) which are frequently sufficient to solve situations involving merely one player and random elements. It is usually considered a distinct subject from gaming or simulation, even though interaction between these subjects often proves beneficial. Game theory also gains insights from conducting game theoretical types of experiments. And since such encounters are laden with various concepts of value and worth, game theory makes frequent

use of utility theory which in its modern form was originally developed as a supplement to the game models.

3. Types of games. It is obvious that the full scope of game theory is extremely broad and most ambitious, and that one must restrict himself to special cases if he is to obtain usable results. There are consequently many logical classifications of competitive situations. Some that have proved helpful are: the number of participants, the number of moves or choices, the constant-sum and general-sum cases, the cooperative and noncooperative situations, the various states of information available to the players, the different restrictions which may be placed on side payments, single play games or multi-stage games in which similar games are played repeatedly, as well as more dynamical situations in which choices are made continuously in time.

4. Forms of a game. In 1944, John von Neumann and Oskar Morgenstern described three abstract models or forms for studying games: extensive form, normal form, and characteristic function form. Most of the models used in the mathematical theory of games are of one of these basic types, or a rather direct variation, extension or generalization of them. A brief technical description of each form will be presented first, and then each type will be described in more detail, along with illustrations and applications.

An n-person game in extensive form can be described by a "rooted tree," that is, a connected graph in the plane with no loops and with a distinguished vertex or node. The distinguished or initial vertex (root) and each additional interior vertex (point, node, locus) corresponds either to one of the n players or to the "chance player." The arcs (edges, lines) ascending from a vertex correspond to the alternate choices this player can make if a play of a game leads to this point. To each terminal vertex there is associated an

n-dimensional vector in which the components represent the payoffs to the respective players. For each node belonging to the chance player there is a probability distribution over the ascending arcs. The state of a player's information at any stage can be described by certain types of subsets of the set of all his vertices which are called information sets. The extensive form usually gives the most complete description of a game and is used to model many game like situations. However, it has not yet proved to be the most suitable form for handling the computational problems involved in actually solving games, except for very simple cases.

An n-person game in normal form is characterized by n sets of integers $N_1, N_2, \ldots, N_n$ (the players' strategy spaces) and a payoff function F from the cartesian product $N_1 \times N_2 \times \ldots \times N_n$ into n-dimensional euclidean space. The vector $F(i_1, i_2, \ldots, i_n)$ describes the respective payoffs to the n players when they each make the single choices $i_1, i_2, \ldots, i_n$. One can also incorporate chance into the game by including a set N_0 of chance moves and a probability distribution over such moves. One then works with expected payoffs in the obvious way. The subject of two-person "zero-sum" games in normal form (the matrix games) is well known, and it has had significant application to many fields. The n-person games can also be viewed as an n-dimensional matrix in which the "rows" in some one dimension correspond to a particular player's choices. The components of the matrix are n-dimensional payoff vectors. Games in normal form are used to study two-person games and n-person "noncooperative" games. In the latter case, the theory of "equilibrium points" is the main solution concept.

An n-person game in characteristic function form is determined by a real valued characteristic function v defined on the set 2^N of all subsets of a finite set N. That is, v assigns the real number $v(S)$ to each subset S

of N. N is the set of n players denoted by 1,2,...,n, and v(S) represents the value (wealth, power) which the coalition S can achieve when its players cooperate. Most models for n-person "cooperative" games are analyzed in this form or in some variation of it.

A game in extensive form can be reduced to a game in normal form by using the game tree to introduce a set of "pure strategies" for each player. Each pure strategy is a choice by the given player of how he will play through the entire game tree in the event of all possible choices by the other players. One then considers "mixed strategies" which are probability distributions over the set of pure strategies. In the case of the two-person zero-sum games, determining "optimal" mixed strategies is mathematically much simpler in this normal form. The optimal mixed strategies in the reduced game (in normal form) can be interpreted in the original extensive form of the game. As for determining optimal strategies, the game in normal form maintains many of the essential strategic features of the original game in extensive form. Historically, there has been little work done on deriving solution techniques directly for games in extensive form.

A game in normal form can be further reduced to a game in characteristic function form. If the game is "constant-sum," then v(S) is the "value" of the coalition S in the resulting two-person constant-sum game between S and its complement N-S. If the game is not constant-sum then v(S) may be taken as the "maximin value" of S in the resulting two-person noncooperative general-sum game between S and N-S, or it may be determined in some other manner. In the former cases the determination of v(S) is rather conservative, because it assumes N-S will cooperate whenever S does. Furthermore, this reduction removes many of the detailed features of the original game: the nature and the timing of the moves, the nature of the information, and the

specific payoffs. One can make several objections to it, especially in the nonconstant-sum case, and any results determined for the game in characteristic function form may have very limited use when interpreted in the normal or extensive form. However, the characteristic function seems most appropriate for studying coalition formation which is an essential feature in the cooperative games. From another point of view, one can begin by assuming that he has a game in characteristic function form independent of the other forms. The value $v(S)$ can be given or derived somehow as the inherent wealth or power of the coalition S when its members cooperate.

5. Historical highlights. Although some game theory concepts have arisen over the past couple of centuries, modern game theory dates from 1944 with the publication of the monumental work Theory of Games and Economic Behavior [25] by von Neumann and Morgenstern. Some particular two-person zero-sum matrix games were analyzed in the early part of the 20th century, and the major theoretical result for these games (the famous Minimax Theorem) was proved by von Neumann in 1928. (See English translation in [24 (1959)].) Although a principle theorem was given by Zermelo in 1913 [29], the first complete definition of a game in extensive form also appeared in Theory of Games and Economic Behavior. However, a slightly different definition due to Kuhn [24 (1953)] is the one most frequently used today. The first game theoretic model for n-person cooperative games was also presented by von Neumann and Morgenstern in 1944 when they presented a theory of solutions (stable sets) based on the characteristic function formulation. Several other solution concepts have since been defined for the cooperative theory. In 1950 Nash [14] generalized the "saddle point" point concept and the minimax theorem for matrix games to the existence of equilibrium points for n-person general-sum noncooperative games in normal form.

An excellent survey of modern game theory until 1957 is presented in the book by Luce and Raiffa [11]. Many books contain an exposition on the matrix games. A current introduction to several major models in the n-person theory appear in Chapters 8, 9 and 10 in the text by Owen [16]. L. S. Shapley and M. Shubik have been preparing an extensive work on the multi-person games and their applications, and some chapters already appear as RAND Corporation reports [21]. Elementary expositions of the n-person theory have been published recently by Davis [4] and Rapoport [18]. A detailed bibliography on game theory until 1959 was compiled by D. M. and G. L. Thompson in [24 (1959)]. The best volume of references and short abstracts of game theory literature to 1968 appears currently only in Russian [26]. Several of the major mathematical papers in the field have appeared in the five issues of the Annals of Mathematics Studies [24]. The introductions to these studies also provide excellent survey material on the subject. A new set of volumes on game theory has recently begun [28]. There is now an International Journal of Game Theory and articles on this subject appear in the new journal Mathematics of Operations Research. A popular book On The Game of Business by McDonald [12] views several real cases from the world of business from a game theoretic point of view.

II. The Normal Form

1. *Introduction*. The simplest game theoretical model is the theory of games in normal form, also referred to as the strategic form or as the matrix (or polymatrix) games. This is simply a list of pure strategies for each one of the players along with a description of the resulting payoffs to the players for any possible choice of strategies by these participants. For the special situation of only two players and when one's loss corresponds to the other's gain, such games can be described by a real valued matrix. This simplest case will be described in some detail. Extensions to general-sum games and those with more than two players will be mentioned later in less detail.

2. *Matrix games*. A *two-person zero-sum finite matrix* game with real payoffs is characterized by an m by n matrix A with real components a_{ij}. Each number (or row) i in $S_I = \{1,2,\ldots,m\}$ corresponds to a *pure strategy* for player I, the *row player*. Each number (or column) j in $S_{II} = \{1,2,\ldots,n\}$ corresponds to a pure strategy for player II, the *column player*. So I has m pure strategies or choices, and I has n pure strategies. The number a_{ij} is the *payoff* from player II to player I if I plays his i^{th} strategy and II plays his j^{th} strategy. If $a_{ij} < 0$ then consider the payoff to be $|a_{ij}|$ from I to II. A *play* of the game consists of I choosing a row and II choosing a column, both in ignorance of the other's choice.

If player I were to pick the i^{th} row, then the worst that could happen to him is that he would obtain $\min_{1 \le j \le n} a_{ij}$. A conservative player I might play a row that corresponds to a maximum of these row minima. This is called a *maximin strategy* for I, and playing this strategy guarantees that he will obtain at least the *lower value* of the game:

$$v_I = \max_{1\leq i\leq m} \min_{1\leq j\leq n} a_{ij}.$$

Similarly define a minimax strategy for player II to be a column which corresponds to a minimum of the column maxima. Playing such a strategy can assure II of losing no more than the upper value of the game:

$$v_{II} = \min_{1\leq j\leq n} \max_{1\leq i\leq m} a_{ij}.$$

These concepts can be illustrated by the following array

I's strategies	II's strategies: 1	2	...	j	...	n	row minima
1	a_{11}	a_{12}	...	a_{1j}	...	a_{1n}	$\min a_{1j}$
2	a_{21}	a_{22}	...	a_{2j}	...	a_{2n}	$\min a_{2j}$
⋮			...				⋮
i	a_{i1}	a_{i2}	...	a_{ij}	...	a_{in}	$\min a_{ij}$
⋮			...				⋮
m	a_{m1}	a_{m2}	...	a_{mj}	...	a_{mn}	$\min a_{mj}$
column maxima	$\max a_{i1}$	$\max a_{i2}$	...	$\max a_{ij}$	...	$\max a_{in}$	v_I / v_{II}

It is easy to prove that $v_I \leq v_{II}$, that is, maximin $\leq$ minimax. As a result of this we can consider two cases: one when $v_I = v_{II}$ and the other when $v_I < v_{II}$.

CASE 1. The Saddle Point Case. If $v_I = v_{II} = v$, then player I can assure himself of obtaining at least the amount v from II by playing a

maximin strategy, and II can hold his losses down to at most v by playing a minimax strategy. Thus v is called the value of the game, and a maximin strategy i_0 and a minimax strategy j_0 are called optimal strategies for I and II respectively. The number v and an optimal strategy for each of the players is called a solution of the game. The game is said to have a saddle point at (i_0,j_0) with value $a_{i_0 j_0} = v$. In this case, prior knowledge of the opponents strategy choice is of no value to the opponent.

One can show that if player I has two optimal strategies i_1 and i_2, and player II has two optimal strategies j_1 and j_2, then $a_{i_1 j_1} = a_{i_1 j_2} = a_{i_2 j_1} = a_{i_2 j_2}$.

CASE 2. The Mixed Strategy Case. Although the most that I can be sure of winning is v_I, it seems that he could "expect" in addition some of the gap $v_{II} - v_I > 0$ if he were to bring in probability considerations and use expected values. A mixed strategy for player I is a probability distribution over his pure strategy set, and it can be described by a probability vector $x = (x_1,x_2,\ldots,x_m)$ where $x_1 + x_2 + \ldots + x_m = 1$ and each $x_i \geq 0$. To play a mixed strategy means to use a random device to pick which pure strategy will be played, the i^{th} pure strategy to be chosen with probability x_i. Similarly, a mixed strategy for player II is a probability vector $y = (y_1,y_2,\ldots,y_n)$ where $y_1 + y_2 + \ldots + y_n = 1$ and each $y_j \geq 0$. Note that a pure strategy can be considered as a special case of a mixed strategy, and each player has an infinite set of mixed strategies.

If player I were to play the mixed strategy x and player II plays his pure strategy j then I would "expect" to obtain

$$E(x,j) = x_1a_{1j} + x_2a_{2j} + \ldots + x_ma_{mj} = xA_j \tag{1}$$

which is the inner (dot, scalar) product of x and the j^{th} column of matrix A which is donated A_j. Likewise, if II plays the mixed strategy y and I plays the pure strategy i, then I expects to obtain

$$E(i,y) = a_{i1}y_1 + a_{i2}y_2 + \ldots + a_{in}y_n = A^i y^T \tag{2}$$

which is the inner product of the i^{th} row of A and y. A^i is the i^{th} row of A and y^T is the transpose of the vector (matrix) y. If I and II play the mixed strategies x and y respectively, then the expected win to I (or loss to II) is

$$E(x,y) = \sum_{j=1}^{n} \sum_{i=1}^{m} x_i a_{ij} y_j = xAy^T.$$

The problem for player I is to find a probability vector x which maximizes the smallest component in the vector xA, i.e., that maximizes the minimum of the numbers of the form (1) as j varies from 1 to n. Let e_t be the t-dimensional row vector with all t components equal to 1. Then I's goal is to find the vector x that maximizes the number v subject to the constraints

$$\begin{aligned} xA &\geq ve_n \qquad (\max v) \\ xe_m^T &= 1 \\ x &\geq 0. \end{aligned} \tag{3}$$

Similarly II's goal is to find a vector y that minimizes the number V subject to

$$(4) \qquad \begin{aligned} Ay^T &\leq Ve_m^T \quad (\min V) \\ e_n y^T &= 1 \\ y &\geq 0. \end{aligned}$$

Note that $v = ve_n y^T \leq xAy^T \leq xVe_m^T = xe_m^T V = V.$

The two optimization problems in (3) and (4) are a special case of a pair of dual linear programs in the theory of linear programming. So a game can be solved by solving a pair of dual linear programming problems. The Minimax Theorem, or Fundamental Theorem of two-person zero-sum matrix games, by von Neumann (1928) (or equivalently, the Duality Theorem in linear programming) states that there exists mixed strategies x_0 and y_0 for I and II, respectively, which solve the problems in (3) and (4), and that $v = V$. In other words there exists probability vectors x_0 and y_0 and a number v such that

$$\begin{aligned} x_0 A &\geq ve_n \\ Ay_0^T &\leq ve_m^T. \end{aligned}$$

The triple (x_0, y_0, v) is called a solution of the game, x_0 and y_0 are said to be optimal strategies, and v is the value of the game.

3. Examples. (i) Consider the following matrix game.

$$\begin{bmatrix} -1 & 1 & 0 & 1 \\ 4 & 2 & 8 & 2 \\ 4 & 0 & 1 & 3 \\ 2 & 1 & 5 & 2 \end{bmatrix}$$

One can compute the row minima, column maxima, lower and upper values as follows:

$$\begin{array}{cccc|l}
\multicolumn{4}{c|}{\begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}} & \begin{matrix} -1 \\ 2 = v_I \\ 0 \\ 1 \end{matrix} \\
4 & 2 & 8 & 3 & \\
 & \shortparallel & & & \\
 & v_{II} & & &
\end{array}$$

Since $v_I = v_{II} = 2 = a_{22} = v$, this game has a saddle point at (2,2) and a value of 2.

(ii) Two athletes tie for the championship and are awarded with a single trophy worth \$6000. They agree to make sealed bids in multiples of \$1000 with the trophy going to the higher bidder and with the winner's bid being paid to the lower bidder. In case of a tie they will flip a coin for the trophy. If we use the expected value in the case of a tie, the game matrix in thousand of dollars is as follows:

$$\begin{array}{c|c}
 & \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\
\begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{bmatrix} 3 & 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 2 & 3 & 4 & 5 & 6 \\ 4 & 4 & 3 & 3 & 4 & 5 & 6 \\ 3 & 3 & 3 & 3 & 4 & 5 & 6 \\ 2 & 2 & 2 & 2 & 3 & 5 & 6 \\ 1 & 1 & 1 & 1 & 1 & 3 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}
\end{array}$$

Each player has two optimal pure strategies, that is, to bid \$2000 or \$3000, and the value of the game is \$3000. The \$2000 bid "majorizes (dominates)" the \$3000 one.

(iii) A brewery worker can buy his beer before he leaves for his summer vacation through the brewery at a discount rate of \$3 per case, whereas it costs \$5 per case at the resort. He has no idea whether there will be cool or warm weather. If it is cool he will drink 10 cases during his vacation, and if it is warm he will drink 15 cases. Any left over is wasted since he drinks only at the brewery when not on vacation and there the beer is free. This game "against nature" has the following matrix.

$$\begin{array}{l} \\ \text{Buy 10 cases} \\ \text{Buy 15 cases} \end{array} \begin{array}{c} \begin{array}{cc} \text{Cool} & \text{Warm} \end{array} \\ \begin{bmatrix} -\$30 & -\$55 \\ -\$45 & -\$45 \end{bmatrix} \end{array}$$

There is a saddle point at (Buy 15 cases, Warm) and the value is -\$45.

(iv) In the matching pennies each player can pick heads H or tails T. Assume player I wins if the coins match and II wins otherwise. The game matrix is

$$\begin{array}{c} \\ H \\ T \end{array} \begin{array}{c} \begin{array}{cc} H & T \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

This game has no saddle point; $v_I = -1$ and $v_{II} = 1$. If I uses the mixed strategy $x = (1/2, 1/2)$ he will expect $E(x,j) = 0$ against either strategy j by II. Player II should use the same strategy. It is easy to check that these strategies are optimal and that the value of the game is 0. A player can effect this mixed strategy by flipping his coin, but he must not let the other know his choice in advance.

(v) The well-known children's game of scissors, paper, stone has the matrix

$$\begin{array}{c c} & \begin{array}{ccc} Sc & P & St \end{array} \\ \begin{array}{c} Sc \\ P \\ St \end{array} & \left[\begin{array}{ccc} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{array}\right] \end{array}$$

The optimal strategies are (1/3, 1/3, 1/3) and the value is 0.

4. Solving matrix games. There are several methods for solving matrix games. A survey of some of these methods is given in Appendix 6 of Luce and Raiffa [11]. It was shown above that the problem of solving a game can be reduced to solving a pair of dual linear programs. This section illustrates how the "primal simplex method" for linear programming can be used to solve matrix games. There will be no detailed discussion of the simplex method in general, but only an illustration of how it works on a couple of specific examples. Those unfamiliar with linear programming can merely note that one proceeds repeatedly from one system of linear equations to another equivalent form until he reaches a system from which the desired solution appears obvious.

In this section we will assume that the game matrix $A = [a_{ij}]$ has $a_{ij} > 0$ for all i and j. This is necessary because we will divide through some inequalities by values which we assumed to be positive. One can prove that if he adds a number a to each entry in the matrix $[a_{ij}]$ to get the matrix $[a_{ij} + a]$, then these games both have the same optimal strategies and their values differ by a. This shows that one can add a constant to each entry in a matrix to obtain a new matrix $A > 0$, and thus v and $V > 0$; and solving this new game yields a solution for the original one by subtracting this constant from its value. One can also multiply all the entries in a game by a positive constant, for example, to clear it of fractions. Since one can prove that the games $[a_{ij}]$ and $[ca_{ij}]$ for $c > 0$ have the same optimal

strategies and that the value of the latter is c times the value of the former.

In section 2 we saw that solving a game was equivalent to solving the dual linear programs in (3) and (4) for x, y and v $(=V)$. In (3) one can divide through by v and define the m-dimensional vector

$$u = \frac{1}{v}x \tag{5}$$

to get

$$\begin{aligned} uA &\geq e_n \\ ue_m^T &= \frac{1}{v} \qquad (\min \frac{1}{v}) \\ u &\geq 0. \end{aligned} \tag{6}$$

Similarly, one can let

$$w = \frac{1}{V}y \tag{7}$$

in (4) and divide by V to get

$$\begin{aligned} Aw^T &\leq e_m^T \\ e_n w^T &= \frac{1}{V} \qquad (\max \frac{1}{V}) \\ w &\geq 0. \end{aligned} \tag{8}$$

The "primal" problem in (8) can be expressed by the following "simplex table"

w_1	w_2	$\cdots$	w_n	w_{n+1}	w_{n+2}	$\cdots$	w_{n+m}	=	
a_{11}	a_{12}	$\cdots$	a_{1n}	1	0	$\cdots$	0	1	
a_{21}	a_{22}	$\cdots$	a_{2n}	0	1	$\cdots$	0	1	
		$\cdots$				$\cdots$		$\cdots$	(min z)
a_{m1}	a_{m2}	$\cdots$	a_{mn}	0	0	$\cdots$	1	1	
-1	-1	$\cdots$	-1	0	0	$\cdots$	0	0	= -z

where all $w_k \geq 0$ and where

(9)
$$z = -\frac{1}{V} .$$

The $w_{n+1}, w_{n+2}, \ldots, w_{n+m}$ are the "slack" variables. In matrix form this table is

w		
A	I_m	e_m^T
$-e_n$	o_m	z

where I_m is the m by m identity matrix and o_m is the m-dimensional vector of all zeros. One can solve the primal problem given in this table for optimal w and z. The final simplex table will also give the value of the optimal u for the dual problem in (6). From (5), (7) and (9), one can then compute the x, y and v which will solve the game A. This method will now be illustrated by some examples.

Example. If one adds 2 to the matrix for Example (v) in section 3 he obtains

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

which has the table

w_1	w_2	w_3	w_4	w_5	w_6		
2	2	1	1	0	0	1	
1	2	3	0	1	0	1	
3	1	2	0	0	1	1	
-1	-1	-1	0	0	0	0	= -z

one can arbitrarily pick the "pivot" element in the first column. The numbers in the last column (but not the last row) are divided by the respective positive elements in this first column, and the minimum occurs for third row. Therefore, the circled number 3 in the third row is the pivot, and "pivoting" on it gives the second equivalent table

0	7/3	-1/3	1	0	-2/3	1/3
0	5/3	7/3	0	1	-1/3	2/3
1	1/3	2/3	0	0	1/3	1/3
0	-2/3	-1/3	0	0	1/3	1/3

The minimum in the last row occurs in the second column, and the minimum of the numbers in the last column divided by the positive numbers in the second column occurs in first row. The next pivot is the circled number 7/3, and pivoting on it gives the third table

0	1	-1/7	3/7	0	-2/7	1/7
0	0	18/7	-5/7	1	1/7	3/7
1	0	5/7	-1/7	0	3/7	2/7
0	0	-3/7	2/7	0	1/7	3/7

The minimum in the last row occurs in the third column, and the minimum of the numbers in the last column divided by the positive numbers in the third column occurs in the second row. The next pivot is the circled number 18/7, and pivoting on it gives the fourth table

w_1	w_2	w_3				
0	1	0	7/18	1/18	-5/18	1/6
0	0	1	-5/18	7/18	1/18	1/6
1	0	0	1/18	-5/18	7/18	1/6
0	0	0	1/6	1/6	1/6	1/2

Since the last row is nonnegative, we have an optimal solution which is

$$w = (1/6, 1/6, 1/6), \ z = -1/2.$$

The optimal solution u for the dual problem can be read off the last row under the slack variables, that is,

$$u = (1/6, 1/6, 1/6).$$

The optimal solution for the game A is

$$x = (1/3, 1/3, 1/3) = y, \quad v = 2.$$

The game

$$A = \begin{bmatrix} 7 & 5 & 6 \\ 0 & 9 & 4 \\ 14 & 1 & 8 \end{bmatrix}$$

is solved by the simplex method in the following tables. From the third table one determines

$$w = (0,\ 1/17,\ 2/17), \quad u = (0,\ 7/68,\ 5/68), \quad z = -3/17.$$

The optimal solution is

$$y = (0,\ 1/3,\ 2/3), \quad x = (0,\ 7/12,\ 5/12), \quad v = 17/3.$$

7	5	6	1	0	0	1
0	9	4	0	1	0	1
14	1	8	0	0	1	1
-1	-1	-1	0	0	0	0

-7/2	17/4	0	1	0	-3/4	1/4
-7	17/2	0	0	1	-1/2	1/2
7/4	1/8	1	0	0	1/8	1/8
3/4	-7/8	0	0	0	1/8	1/8

0	0	0	1	-1/2	-1/2	0
-14/17	1	0	0	2/17	-1/17	1/17
63/34	0	1	0	-1/68	9/68	2/17
1/34	0	0	0	7/68	5/68	3/17

5. Applications. The finite matrix games, along with their generalizations to the cases of infinitely many pure strategies and repeated play of such games, have proved useful in a great number of applications. These include allocation of resources (games of partitioning), duel theory (games of timing), search

problems, as well as statistical decision theory. The theory of two-person, zero-sum games in normal form with finitely many strategies is mathematically equivalent to the theory of linear programming and consequently has as many uses. Two extremely simple examples will be given in this section to illustrate such applications.

(i) A simple search game. Assume that there are n points on a line labeled (in order) $1,2,\ldots,n$. Player I picks one point at which to hide. Player II picks points, one at a time, at which to look for I. After each choice of a point, II is told whether he has found I, whether he has chosen a point too high (i.e., to the "right" of I), or whether he has chosen a point too low (i.e., to the "left" of I). The payoff to player I (from II) is the number of choices II must make in order to find I.

For player I there are n strategies: $1,2,\ldots,n$; i.e., the n places at which he can hide. A strategy for player II is a search plan or hunting strategy. This can be represented by an n-tuple with a "1" in the position of his first look, with a "2" to both the right and the left of the "1" which tells where his second look will be, with a "3" on both sides of each "2" (so that the "1" falls in the middle of the four "3's") which tells where his third look will be, etc. For example, if $n = 7$, then the hunting strategy

$$(2,3,1,3,4,2,3)$$

says that II's first look is in position 3, that if he is too high then his second look is in position 1 and if he is too low (on his first look) then his second look is in position 6, etc.

This game has been solved by S.M. Johnson for $n \leq 11$ in [24 (1964)]. The solution for the case when $n = 3$ follows. When $n = 3$, player I can pick position 1, 2 or 3 in which to hide. Player II has five hunting strategies:

(1,2,3) (1,3,2) (2,1,2) (2,3,1) (3,2,1).

The game matrix is

$$
\begin{array}{ccc}
 & & \text{II} \\
 & & \\
\text{I} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 1 & 2 & 2 & 3 \\ 2 & 3 & 1 & 3 & 2 \\ 3 & 2 & 2 & 1 & 1 \end{bmatrix}
\end{array}
$$

Note that the payoffs in each column give the same triple as the corresponding strategy for II. This game is solved in the following simplex tables. The optimal strategy for I is (.4,.2,.4) and for II it is (0,.2,.6,.2,0). The value is 9/5.

1	1	2	2	3	1	0	0	1
2	3	1	3	2	0	1	0	1
3	2	2	1	1	0	0	1	1
-1	-1	-1	-1	-1	0	0	0	0
1/2	1/2	1	1	3/2	1/2	0	0	1/2
3/2	5/2	0	2	1/2	-1/2	1	0	1/2
2	1	0	-1	-2	-1	0	1	0
-1/2	-1/2	0	0	1/2	1/2	0	0	1/2
-1/2	0	1	3/2	5/2	3/2	0	-1/2	1/2
-7/2	0	0	9/2	11/2	2	1	-5/2	1/2
2	1	0	-1	-2	-1	0	1	0
1/2	0	0	-1/2	-1/2	0	0	1/2	1/2
2/3	0	1	0	2/3	5/6	-1/2	1/3	1/3
-7/9	0	0	1	11/9	4/9	2/9	-5/9	1/9
11/9	1	0	0	-7/9	-5/9	2/9	4/9	1/9
1/9	0	0	0	1/9	2/9	1/9	2/9	5/9

(ii) A duel. Consider a finite duel in which each player has a gun which contains one bullet. Player I has a silent gun, i.e., II does not know when I has fired; and player II has a noisy gun, i.e., I knows if II has fired. At the start the players are 10 steps apart. Assume that both players have the same accuracy function: after a player's k^{th} step his probability of hitting his opponent is $k/5$ where $k = 0,1,2,3,4,5$. Any player who hits his opponent collects one unit from this opponent, and the duel then terminates.

If player I shoots after his i^{th} step and player II shoots after his j^{th} step, then the payoffs (in expected values) from II to I are:

$$a_{ij} = \frac{i}{5} - (1 - \frac{i}{5})\frac{j}{5} \quad \text{when } i < j$$

$$a_{ij} = 0 \quad \text{when } i = j$$

$$a_{ij} = (1 - \frac{j}{5}) - \frac{j}{5} \quad \text{when } i > j.$$

The game matrix (multiplied by 25) is given below.

	i \ j (II)	0	1	2	3	4	5
	0	0	-5	-10	-15	-20	-25
	1	25	0	-3	-7	-11	-15
	2	25	15	0	1	-2	-5
I	3	25	15	5	0	7	5
	4	25	15	5	-5	0	15
	5	25	15	5	-5	-15	0

The third row $(i = 2)$ "dominates" the first and second row, and the fourth row dominates the sixth row. These dominated rows can be deleted without losing all of the optimal solutions for this game. Similarly, the first two columns can be deleted because they dominate the third column. It is sufficient to solve the game:

i \ j	2	3	4	5
2	0	1	-2	-5
3	5	0	7	5
4	5	-5	0	15

Adding 5 to each entry in the matrix gives the matrix

$$B = \begin{bmatrix} 5 & 6 & 3 & 0 \\ 10 & 5 & 12 & 10 \\ 10 & 0 & 5 & 20 \end{bmatrix}$$

The latter game B is solved in the simplex tables below.

5	6	3	0	1	0	0	1
10	5	12	10	0	1	0	1
10	0	5	20	0	0	1	1
-1	-1	-1	-1	0	0	0	0
5	6	3	0	1	0	0	1
5	5	19/2	0	0	1	-1/2	1/2
1/2	0	1/4	1	0	0	1/20	1/20
-1/2	-1	-3/4	0	0	0	1/20	1/20
-1	0	-42/5	0	1	-6/5	3/5	4/10
1	1	19/10	0	0	1/5	-1/10	1/10
1/2	0	1/4	1	0	0	1/20	1/20
1/2	0	23/20	0	0	1/5	-1/20	3/20
-5/3	0	-14	0	5/3	-2	1	2/3
5/6	1	1/2	0	1/6	0	0	1/6
7/12	0	19/20	1	-1/12	1/10	0	1/60
5/12	0	9/20	0	1/12	1/10	0	11/60

From the last table one sees that the game B has value $v_B = 60/11$ and optimal strategies $(1/12, 1/10, 0)(60/11) = (5/11, 6/11, 0)$ and $(0, 1/6, 0, 1/60)(60/11) = (0, 10/11, 0, 1/11)$ for I and II respectively. It follows that the original game matrix has value $v = v_B - 5 = 5/11$ and optimal strategies $(0, 0, 5/11, 6/11, 0, 0)$ and $(0, 0, 0, 10/11, 0, 1/11)$ for I and II respectively. Since the original game matrix was obtained by multiplying by 25, one gets that the value of the duel is 1/55. Player I shoots with probabilities 5/11 and 6/11 after steps 2 and 3. Player II shoots with probability 10/11 after step 3, and with probability 1/11 he walks right up to I where he has a sure shot (if I has not already killed II).

In many duels and search problems the players have an infinite number of pure strategies, for example, all the points on an interval, or all the continuous curves in the plane. Examples of such duels are given in [5] and [6].

6. Two-person, general-sum games. A two-person general-sum game in normal form with a finite number of strategies is characterized by an m by n payoff matrix in which each entry is a pair of numbers corresponding to the payoffs to the two players. Such games can likewise be described by a pair of real valued matrices A and B, and hence are referred to as the bimatrix games. The rows and columns respectively correspond to the pure strategies for the players I and II. One can also consider mixed strategies for each player which are probability distributions over his set of pure strategies.

The best known example of general-sum games is the two-person prisoner's dilemma, due originally to A. W. Tucker in 1950 and since studied in great detail by Anatol Rapoport [19] and many others. Two men, I and II, are charged with a joint crime, and held separately by the police. Each is told that

(a) if one confesses and the other does not, then the former will receive a reward of one unit and the latter will be fined two units, and

(b) if both confess, each will be fined one unit.

The prisoners know however that

(c) if neither confesses, both will go free.

The resulting payoff matrix follows.

		II	
		Confess	Not Confess
I	Confess	(-1,-1)	(1,-2)
	Not confess	(-2,1)	(0,0)

The first component in each payoff vector is for player I whereas II obtains the second component. For each man, the strategy "Confess" dominates the strategy "Not confess." If played noncooperatively, both players are likely to confess, resulting in each being penalized one unit. Technically, the payoff (-1,-1) is the unique "equilibrium" point in this game. On the other hand, if the players are allowed to play cooperatively, i.e., to communicate (perhaps via lawyers) and form binding agreements, then it is likely that they will not confess and both will go free. The latter outcome which appears on the northeast edge of the "payoff space" is clearly the best solution from a global perspective.

Repeated play of the prisoner's dilemma games provides a miniature model of what happens in arms races, price wars, run-away advertising campaigns, and the "tragedy of the commons," in which each individual citizen uses more than his share of a renewable resources, thus causing its ultimate depletion. One obtains an interesting three-person game if he adds the State (or world community) to the above prisoner's dilemma, where the State's payoff depends upon the nature of the fines assessed.

When a game is general-sum or involves more than two players, then it is essential to distinguish between whether this game is played cooperatively or noncooperatively. A cooperative game of two players involves elements of both cooperation and competition. The players normally agree to restrict the outcomes to "Pareto optimal" ones, i.e., payoff vectors on the "northeast"

boundary of the set of all potential payoffs. They then must bargain over which payoff on this boundary is a distribution of wealth acceptable to both parties. This latter settlement is noncooperative in the sense that increasing one player's payoff decreases the other's outcome. Several methods for determining an equitable payoff vector from the Pareto optimal ones have been proposed. Chapter 6 in the book by Luce and Raiffa [11] is still an excellent discussion of many such cooperative, or bargaining solution, concepts. When there are more than two players in a cooperative game, then the theory changes drastically, since coalition formation then becomes a major aspect of the problem. Such multiperson cooperative games will be discussed in Part III.

7. n-person noncooperative games. For noncooperative games in normal form, the main solution concept is that of an equilibrium strategy. Assume that each of the n player ($n \geq 2$) has chosen a particular strategy (pure or mixed). This strategy n-tuple is in equilibrium if no player can improve his payoff by he alone switching his strategy choice. That is, a unilateral deviation by just one player cannot improve his situation, if the other players continue to play these particular strategies. The strategy pair (Confess, Confess) is an equilibrium outcome in the prisoner's dilemma game. The optimal strategies in a two-person, zero-sum game are precisely the equilibrium outcomes.

John F. Nash, Jr. [14,15] proved that every game with a finite list of pure strategies for each player must have at least one equilibrium n-tuple in mixed strategies. The problem of actually determining such equilibrium strategies corresponds to optimizing a multilinear function with inequality constraints. Precise algorithms exist for the case of two players, and recent algorithms for approximating fixed point values are often used currently for larger values of n.

Equilibrium points are important in applications of game theory to problems in fields such as economics and operations research where they may correspond to prices in an economic market, optimal bidding strategies, or the best allocation of vehicles over a road network.

One can read more about the theory of equilibrium points for noncooperative games in the books by Burger [2] and by Parthasarathy and Raghavan [17].

III. The Characteristic Function Form

1. Introduction. The first detailed game theoretical model for multiperson cooperative games was presented in 1944 by J. von Neumann and O. Morgenstern, and is called the cooperative, coalitional, or characteristic function form of a game. This model, which comprises the greatest part of their text [25], is employed in the study of situations involving three or more participants acting in a cooperative mode. In such cooperative behavior the agents are allowed to get together in coalitions and to undertake joint action for the purpose of mutual gain.

The estimated worth of any potential coalition is a crucial aspect of such models, and this value is expressed as a numerical measure by what is called the characteristic function. The term von Neumann-Morgenstern solution or simply solution in this context refers to the final solution concept or end product for their coalition formulation of cooperative games. A brief intuitive and verbal description of their model is presented first, and then a more precise technical definition is given.

2. An overview of the model. There are four essential concepts or fundamental definitions which make up the basic cooperative model. First, a multiperson game is characterized by merely assigning a real number to each possible coalition of the n participants, where each number represents the value, wealth or power achievable by this coalition when its members cooperate. Next, one describes a set of n-dimensional payoff vectors called imputations. This presolution set represents all reasonable or realizable ways of distributing the available wealth among the n parties. Then, one introduces a preference relation between certain pairs of imputations. One imputation is said to dominate another, if there is some coalition in which each of its

members prefers the former payoff to the latter one, and if this coalition as a whole is not obtaining more in the former imputation than it can effectively realize in the game through its own efforts. Finally, a von Neumann-Morgenstern solution is any subset of the imputation space which possesses a certain internal and external consistency; namely, no imputation in a solution dominates another one in this solution, and any imputation not in the solution is dominated by another one within the solution. So a solution when taken as a whole is preferred to precisely those imputations outside of this set. That is, a solution set is the complement of its "dominion."

3. The basic model. An n-person game (in characteristic function form) consists of a pair (N,v) where $N = \{1,2,\ldots,n\}$ is a set of n players labeled by $1,2,\ldots,n$, and where v is a characteristic function which assigns a real number $v(S)$ to each nonempty subset (or coalition) S of N. (Note that n again represents the number of players in the game, rather than the number of strategies for player II as was the case in Part II.) One normally takes the integer $n \geq 3$, and assumes that v is superadditive, i.e., $v(S \cup T) \geq v(S) + v(T)$ for disjoint subsets S and T; but this latter assumption is not necessary for much of the theory. The set of imputations, denoted by A, consists of all vectors x such that $x_1 + x_2 + \ldots + x_n = v(N)$ and $x_i \geq v(\{i\})$ for all i in N, where each x_i is a real number and represents a payoff to player i. These two relations are referred to as Pareto optimality and individual rationality, respectively. An imputation x dominates an imputation y whenever there is some coalition S such that $x_i > y_i$ for all i in S and $\sum_{i \in S} x_i \leq v(S)$. When this latter condition is satisfied one says that x is effective for S. A subset V of A is called a von Neumann-Morgenstern solution if it satisfies two

conditions: no imputation x in V dominates any imputation in V; and every imputation y not in V is dominated by at least one imputation in V. These conditions are called internal stability and external stability, respectively, and now one frequently refers to such a solution as a stable set.

4. An example. An illustrative example, known as the three-person "constant-sum" game, is given by $N = \{1,2,3\}$, $v(\{1,2,3\}) = v(\{2,3\}) = v(\{1,3\}) = v(\{1,2\}) = 1$, and $v(\{i\}) = 0$ for $i = 1, 2$ and 3. Intuitively, any coalition of two or three players has the power to distribute the total value 1 among the three players, e.g., a decision by majority rule; whereas an individual player by himself is assured of only his value 0. The imputation space consists of the triangular set or simplex $A = \{(x_1,x_2,x_3)$: $x_1 + x_2 + x_3 = 1$ and $x_i \geq 0$ for $i = 1,2,3\}$. One solution, which has only three imputations as well as the "symmetry" of the game, is the set $V = \{(0,\frac{1}{2},\frac{1}{2}), (\frac{1}{2},0,\frac{1}{2}), (\frac{1}{2},\frac{1}{2},0)\}$. Any other solution for this game consists of all those imputations which give a fixed amount $x_i = c_i$, with $0 \leq c_i < \frac{1}{2}$, to one player i, and distributes nonnegative amounts x_j and x_k to the other two players so that $x_j + x_k = 1 - c_i$; and each of these latter sets is referred to as a discriminatory solution.

5. Nature of the theory. To develop a mathematical theory for a given game model like the one above, one searches for answers to questions such as those about the existence, uniqueness, nature, mathematical properties, structure, and various representations and characterizations for the ensuing solution concepts. From the applied point of view one is concerned with the computability and interpretation of solution sets, as well as with whether they provide any new insights into theory, or in practice when applied to situations in the real world.

It was finally demonstrated by W. F. Lucas [8,9] that not every game need have a solution. However, all such known examples have two or more players, and it appears that the nonexistence of solutions is rare. On the other hand, some solution sets have been characterized for several special but large classes of games. A good sample of the research in this direction is contained in several papers in four of the five volumes devoted to game theory in the Annals of Mathematics Studies edited by A. W. Tucker and others [24]. It turns out that many games have a great abundance of different solution sets. This lack of uniqueness, plus the great multiplicity of imputation in most individual solutions, leaves a great arbitrariness or ambiguity in determining any ultimate payoff vector for many games. Furthermore, some solutions are known to have a very complex or elaborate structure as is known from work of L. S. Shapley, and it is unlikely that one will be able to give a plausible intuitive or behavioralistic interpretation to all such irregular sets, or to design algorithmic methods for constructing them. It appears in general that solution theory is a deep and difficult mathematical subject and that many rich and fascinating structures are possible. A very readable introduction to solution theory as well as the most complete bibliography on the subject will appear in Chapter 6 of a forthcoming book by L. S. Shapley and M. Shubik; and a preliminary version by the authors [21] has appeared in report form. A more technical survey of some important recent results on this subject was written by Lucas [10].

Several generalizations, variations and extensions of the classical von Neumann-Morgenstern theory of solutions have also been investigated. These include in part: the games without side payments surveyed by R. J. Aumann in [22] and included in Shapley and Shubik [21], the games in partition function form presented by R. M. Thrall and Lucas [23], more dynamical approaches

to solution concepts as discussed in the work of R. J. Weber [27], the game models which make use of an infinite number of players, and models built upon other abstract mathematical structures. In addition, several different types of models and alternate solution concepts along the lines of the classical formulation for games in coalition form have since been introduced. The core, value theories, the bargaining sets and the nucleolus for cooperative games are discussed in the next section.

6. Other solution concepts. Several other solution concepts have been developed since the original von Neumann-Morgenstern solutions (or stable sets). Four of these alternate solution ideas, which are well known and have use in applications, will be introduced in this section. These models make use of the same definition for a game (N,v) and for the set of imputations A as those given above.

(i) The core. The concept of the core of a game was defined explicitly by D. B. Gillies in [24] and L. S. Shapley in 1953. The core of the game (N,v) is the set

$$C = \{x \in A: \textstyle\sum_{i \in S} x_i \geq v(S) \text{ for all } S \subset N\}.$$

No coalition can overturn an agreed upon imputation in the core, since no element in A can dominate an imputation in C. For superadditive games, the core is precisely those imputations which are maximal with respect to the relation of dominance.

Consider the three-person game with $v(\{1,2,3\}) = v(\{1,2\}) = v(\{1,3\}) = 0$ and $v(\{2,3\}) = v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$. For this example, C consists of the one imputation $(1,0,0)$ which gives the full amount to player 1. In this "veto-power" game, C treats 1 as though he were a "dictator".

A main problem with the core is that for many games it turns out to be the empty set. This is true, for example, in the three-person, constant-sum game in section 4. In applications of game theory to political science, many games have an empty core. Whereas, many examples of games which arise in economics do have a nonempty core, and the core then proves to be an important analytical tool. However, the example in this section illustrates that a nonempty core may sometimes prove to be "too small," and not to contain the outcomes which would likely arise from actually playing the game.

(ii) The Shapley value. In 1953 Shapley [24] introduced a solution concept which determines a unique imputation ϕ for each game (N,v). This outcome is now called the Shapley value and is given by the formula

$$\phi_i = \sum \frac{(s-1)!(n-s)!}{n!} [v(S) - v(S - \{i\})]$$

for each $i \in N$, where s is the number of players in coalition S and the summation is taken over all subsets S of N which contain the player i. The value ϕ is the unique imputation which satisfies four axioms stated by Shapley which are called efficiency, the dummy axiom (i.e., powerless players receive nothing), symmetry, and additivity. These axioms are properties which would be desirable of any equitable allocation scheme. The Shapley value can thus be viewed as a fair-division solution concept. It has been used in economic problems to set fair rates for services, and in political science to assign power equitably. It also has a probabilistic interpretation as the average incremental gain caused by player i joining coalition $S - i$ to obtain S. Furthermore, ϕ also arises from certain fair bargaining schemes, i.e., where players split equitably any gains from all potential ways of building up from n separate players to the grand coalition N.

(iii) The nucleolus. Another solution concept which always gives a unique imputation for any game (N,v) is the nucleolus which was introduced by D. Schmeidler [20]. First, define the excess of any nonempty subset S of N with respect to the imputation $x \in A$ as $e(x,S) = v(S) - \sum_{i \in S} x_i$. One can set $e(x,\emptyset) = 0$ for the empty set $\emptyset$, and note that $e(x,N) = 0$. The excess represents the "group attitude" of the coalition S towards its payoff at imputation x. In the core where each coalition S is satisfied, $e(x,S) \leq 0$. If S gets less than its value $v(S)$ at some imputation x, then the corresponding excess is negative and S has incentive to form and ask for more. The nucleolus is the imputation which minimizes the largest excess, i.e., it simultaneously minimizes the largest coalitional "complaint." In the event that this maximum excesses attains a minimum at various imputations, then one compares the next largest excesses, and so on, in order to arrive at the unique nucleolus. In other words, for each x one arranges the excesses $e(x,S)$ in nonincreasing order. The nucleolus is then the one particular x which gives a minimum in the "lexicographical" ordering of these arrangements.

The nucleolus always exists and is unique for each game. It is an element in the core when the latter is nonempty, and it is also a member of many, but not all, of the bargaining sets mentioned below. It can be computed by solving a sequence of linear programming problems or equivalently one very large program, but such computations are lengthy unless n is small. In applications, the nucleolus can also be used to set rates for services and appears to be a good approach for setting tax rates. It has been proposed as a scheme for setting fines or other charges for firms that pollute the environment.

(iv) Bargaining sets. In 1964 R. J. Aumann and M. Maschler published in [24] some dozen different solution concepts referred to as bargaining sets.

Their models imitate rather closely what actually takes place in real bargaining situations, and are similar to what hapens in game theoretical experiments. They also consider more explicitly how the set N of players fragments into coalition structures, i.e., into partitions $P = \{P_1, P_2, \ldots, P_m\}$ of N, where the P_i are disjoint coalitions whose union is N. They also enlarge the set of imputations to include more vectors than given by A. Every such extended imputation x along with a corresponding coalition structure P will be called a payoff. A payoff (x,P) is stable, or is an element of a bargaining set, if it can be "defended" against various other payoffs which will be proposed instead.

The crucial concepts in their models are the relations of objection and counterobjection. These technical definitions are rather lengthy, and thus only intuitive statements about them are presented here. An objection by a coalition K to a disjoint coalition L at a payoff (x,P) is a second payoff (y,Q) in which each player in K obtains more and in which each player they must use to achieve this new payoff does at least as well. A counterobjection by L to K in this objection is a third payoff (z,R) in which the players in L obtain at least their original amounts in x and in which any "partners" in R whose cooperation L needs are not worse off than they were in x or y. A payoff is stable if for each objection to it there exists a corresponding counterobjection. A bargaining set is the set of those payoffs which are stable. Since there are different ways to define payoff, objection and counterobjection, there are several different solution concepts which are called bargaining sets.

So a payoff is not unstable merely because some group can object to it. Many such objections really carry little weight, since they can easily be countered. A payoff is unstable, however, if there is any objection to it which cannot be countered.

7. <u>An application to politics</u>. A game (N,v) in characteristic function form is called a <u>simple</u> game if $v(S) = 0$ or 1 for each coalition S contained in the player set N. A simple game is said to be <u>monotone</u> if $S \subset T$ implies $v(S) \leq v(T)$. Monotone simple games can be used to model many voting situations: a <u>losing</u> coalition S has value $v(S) = 0$ and a <u>winning</u> coalition S is assigned value $v(S) = 1$. A particular class of such voting games is the <u>weighted voting</u> games in which each player i in N casts a ballot with weight w_i and in which a coalition S wins whenever its total weight $\sum_{i \in S} w_i$ equals or exceeds some given number q called the <u>quota</u>.

The value type of solution concepts, such as the Shapley value, can be used as indices to measure the voting powers of the various players. One such value concept, proposed by the lawyer John F. Banzhaf, III [1], has actually been accepted in several court rulings in the United States as a reasonable measure of power for determining equity in certain weighted voting systems.

As an illustration, consider a corporation with five stockholders denoted by 1, 2, 3, 4 and 5 who own 40, 25, 20, 10 and 5 percent of the stock respectively. Assume that a simple majority is necessary to pass any legislation. The minimal sized winning coalitions are $\{1,2\}$, $\{1,3\}$ and $\{2,3,4\}$. The Shapley value for this game is the vector $\phi = (5,3,3,1,0)/12$, and it can be defended as a reasonable measure of the respective voting powers for the players. Note that player 5 is powerless; if he alone switches his vote he will never change the outcome on any ballot.

8. <u>An application to economics</u>. In a pure exchange economy each trader brings a bundle of commodities to a market, and these participants proceed to redistribute their goods. Presumably the goal of each such economic agent

is to depart with a commodity bundle which he prefers over his initial one. Assume furthermore that each such trader has a utility function which expresses his preferences over the possible bundles which he can secure, and assume that each one attempts to maximize his resulting utility. This situation can be treated as a multiperson cooperative game in which the characteristic function value $v(S)$ of each coalition S is the maximum total utility to the players in S if they achieve an optimal trade with those goods available within their coalition. To simplify our model, assume that the value to coalition S is the sum of the individual utilities of the players in S for the bundles they each receive.

As a simple illustration, consider a friendly coffee break with three participants denoted by 1, 2 and 3. Each participant brings certain ingredients and each has a different preference for what he wishes to drink. Let the commodities be coffee, tea, cream and sugar. Player 1 has two units of coffee, and prefers to drink tea with sugar. Player 2 has one unit of tea and wishes to drink coffee with cream. Player 3 has two units of cream and three units of sugar and desires to drink coffee with cream and sugar. One can describe a typical commodity vector as $y = (y_1,y_2,y_3,y_4)$, and denote the initial bundles for the three players by $w^1 = (2,0,0,0)$, $w^2 = (0,1,0,0)$ and $w^3 = (0,0,2,3)$. And their respective utility functions will be assumed to be $u^1(y) = \max\{y_2,y_4\}$, $u^2(y) = \max\{y_1,y_3\}$, and $u^3(y) = \max\{y_1,y_3,y_4\}$. Then the total utility available to a coalition $S \subset \{1,2,3\}$ is $\max \sum_{i\in S} u^i(y^i)$ where y^i is player i's bundle and where this maximum is subject to the constraint, or conservation law, $\sum_{i\in S} y^i = \sum_{i\in S} w^i$. Under these assumptions, the characteristic function for this game is $v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{1,2\}) = v(\{2,3\}) = 0$, $v(\{1,3\}) = 2$, and $v(\{1,2,3\}) = 3$. The set of imputations is $A = \{(x_1,x_2,x_3)\colon x_1 + x_2 + x_3 = 3;\ x_1,\ x_2$ and $x_3 \geq 0\}$,

and the core and unique stable set is $C = V = \{x \in A: x_1 + x_3 \geq 2\}$. No distribution x in C can be improved upon by some coalition which decides to go it alone.

IV. Games in Extensive Form

1. Graph theory. A few very simple ideas from graph theory will be mentioned briefly, since a game in extensive (or tree) form is usually defined in terms of such concepts. A graph consists of a set of points, called vertices or nodes, along with a set of (unordered) pairs of distinct vertices, called edges or arcs. The edges can be viewed as lines connecting the corresponding pairs of vertices. The degree of a vertex is the number of distinct edges in which it appears, i.e., the number of lines emanating from this vertex. A graph has a cycle if one can "pass" through a sequence of vertices and return to the original one by means of distinct edges. Such a closed path will not retrace any of its edges. A graph is connected if one can pass from any vertex to any other one by means of a sequence of adjacent edges.

2. Definitions. A general n-person game in extensive form is a topological tree (i.e., a finite connected graph with no cycles and with no vertices of degree two) with the following specifications. There is one distinguished vertex corresponding to the starting point, and referred to as the root. Each nonterminal vertex is a choice point and is labeled with one of the n players $1,2,\ldots,n$; or by the "chance player" denoted by 0. Each edge "leading out" of a vertex describes a possible move by the corresponding player if this point in the game is reached. Each vertex labeled by 0 has a probability distribution over its moves. To each terminal vertex there is assigned an n-dimensional payoff vector whose components describe the outcomes to the respective players when the game ends there. The set of all vertices for a particular player i is partitioned into information sets. When it is one of his turns to make a choice, i is aware of the information set he is in, but he does not know the precise vertex within this set. For each vertex in a

given information set there must be the same number of potential moves, and one normally assumes that in playing a game no such vertex can come "after" another one in the same information set.

When engaged in a particular game each player is faced with the problem of how to best play the game in order to maximize his expected payoff. A player's complete plan for playing a game is called a strategy. There are several different ways in which one can specify such a coherent plan of action. A pure strategy for player i is a rule for picking a particular move at each of his information sets. It is a function which assigns to each information set a particular one of his available choices. A mixed strategy for i is a global randomization over his pures, that is, a probability distribution over the set of all pure strategies. A behavioral strategy for i is an overall plan for local randomization; that is, a behavioral strategy consists of a class of probability distributions such that one particular distribution is assigned to the set of moves at each one of his information sets. Certain combinations of "partial" pure strategies called "signaling strategies" and "associated" behavioral strategies are called composite strategies.

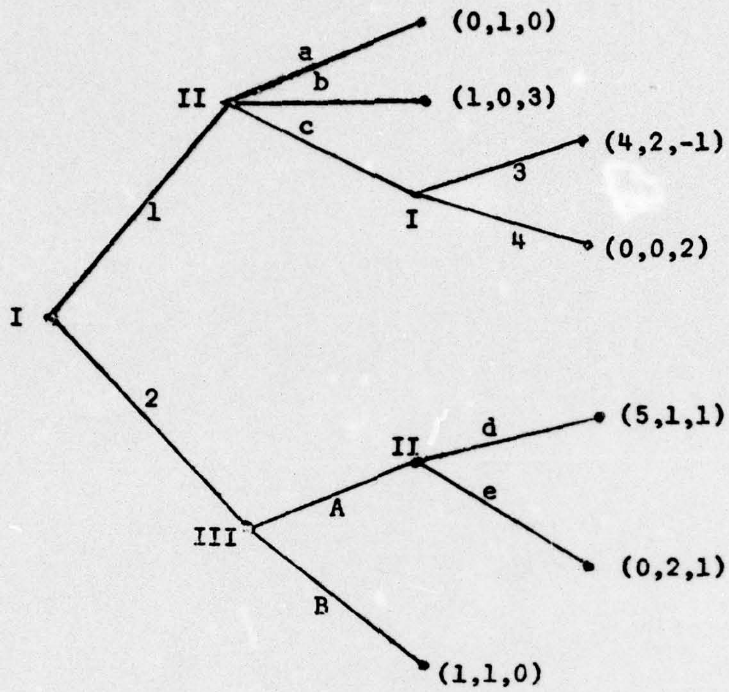
The main solution concept for games in extensive form is the same Nash equilibrium point which was introduced in Part II, i.e., a collection of n optimal strategies (one for each player) such that a unilateral change in strategy by only one player cannot possibly improve his expected payoff. The principal computational problem is to determine such equilibrium points for a given game.

3. Information and existence theorems. Most of the major results about extensive games are of a theoretical nature. These theorems guarantee the existence of optimal strategies which will achieve an equilibrium. In 1912

E. Zermelo [29] demonstrated the existence of an optimal pure strategy for two-person zero-sum games with perfect information, that is, games such as checkers or chess, in which all information sets contain a single vertex. A player has perfect information implies that he can always reconstruct the entire past history of the game along the unique path from the initial vertex (root) up to his current choice vertex. H. W. Kuhn [7, 24 (1953)] extended these results about optimal pure strategies to the n-person general-sum games with perfect information. In 1928 John von Neumann showed the existence of optimal mixed strategies for any two-person zero-sum game; this is the Minimax Theorem or Fundamental Theorem of Game Theory mentioned before. Nash [14,15] extended the equilibrium concept beyond the two-person zero-sum games, and he proved that there are optimal mixed strategies in the n-person general-sum case. Kuhn also showed the existence of optimal behavioral strategies for games with "perfect recall" such as in poker. A game has perfect recall if each player is "aware" at each of his moves of precisely what moves he chose prior to it, but may not know all the prior choices made by the other players. G. L. Thompson [24 (1953)] proved that composite strategies suffice to obtain an equilibrium point for an arbitrary game; and he illustrated his theory for simplified models of bridge in which partners are treated as single players who alternately "forget" certain information.

4. Examples. The following examples will illustrate some of the definitions and theorems presented above.

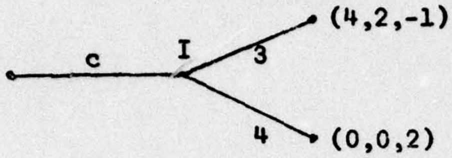
(i) Consider the following three-person game with perfect information in which the players are denoted by I, II and III.



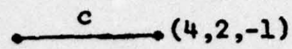
Player I begins by choosing either the upper branch labeled 1 or the lower branch labeled 2. In the former case, II then picks a, b or c. Choice a or b will end the game with the payoff vectors (0,1,0) and (1,0,3) respectively. Whereas choice c leaves the final move up to player I who then picks 3 or 4 to terminate at outcomes (4,2,-1) or (0,0,2). If I had initially chosen the lower branch 2, then player III chooses between moves A and B. Move B ends the game, whereas choice A leaves the final move to player II who picks either d or e. The equilibrium payoff (4,2,-1) is obtained when player I, II and III play their <u>pure</u> strategies (1,3), (c,e) and A, respectively.

The proof that a (finite) game with perfect information has a pure strategy optimal solution follows from a simple backwards induction, i.e.,

the game tree can be repeatedly "pruned" back from the terminal vertices to the root. In our example, the branch

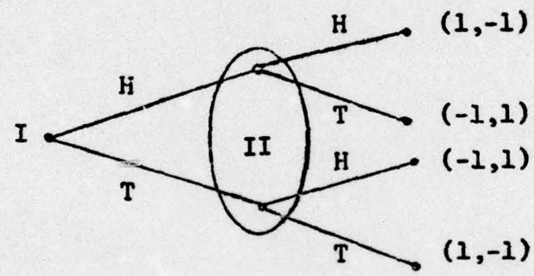


can be replaced by



since a rational player I will surely use alternative 3 instead of 4. One can work back in this way from the terminal nodes to the initial one.

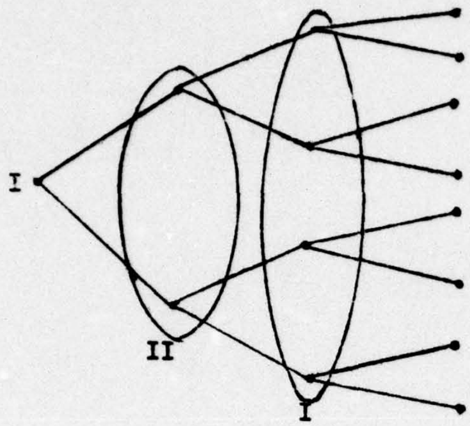
(ii) The well known two-person game of matching pennies can be described as follows.

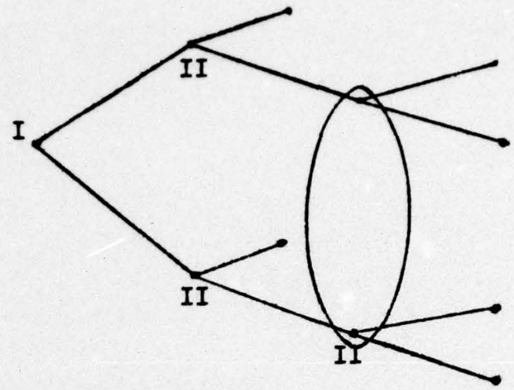


Player I picks either heads H or tails T, and player II has the same choices H and T. Player I wins if the coins match and II wins if they do not match. The information set which incloses the two vertices for II indicates that he is unaware of I's choice, e.g., I and II may in practice move simultaneously. Optimal mixed (or behavioral) strategies for this game are $(\frac{1}{2},\frac{1}{2})$ and $(\frac{1}{2},\frac{1}{2})$, i.e., each plays H and T with equal probability. The value of the game is 0, which indicates that is is a <u>fair</u> game. In this game, player II is not sure of where he is at because of an unknown move by I, and not because he forgot his own choice at some prior vertex.

(iii) The following two simple examples indicate how one can forget what he had done or known at an earlier stage of a game. One who is unable to so reconstruct <u>his</u> past history of play when at one of his information sets does

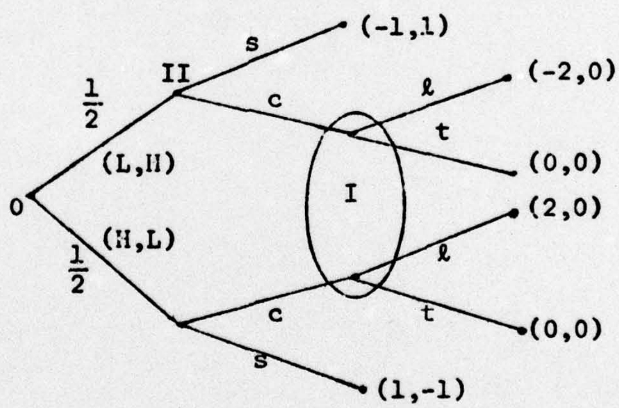
not have perfect recall. The payoffs are not listed in these examples.





In the first example I forgets what action he took at an earlier stage, and in the second example II forgets what he knew at the previous move.

(iv) The following example, given by H. W. Kuhn, indicates how behavioral strategies need not suffice to achieve the best solution in a game without perfect recall.



Chance, denoted by 0, deals a high card H to one player and a low card L to the other, with each deal being equally probable. The holder of the high card then receives 1 unit from the other player, and also has the choice of whether to stop the game s or to continue it c. In the latter case player I has the option of leaving the cards the way they are, ℓ, or of trading cards

with his opponent, t. However, at this stage of the game I has forgotten who originally held the high card.

The optimal mixed strategy for player I is to mix his pure strategies (s,t) and (c,ℓ) with equal probability; and II should play both s and c with probability $\frac{1}{2}$. An expected payoff to player I of $\frac{1}{4}$ will result. One can show however that if player I were restricted to a behavioral strategy, i.e., one probability distribution over c and s and another over ℓ and t, then the best expected value which he can assure himself is 0.

It is true that each mixed strategy induces a behavioral strategy in a natural way, and each behavioral strategy induces a particular mixed strategy which in turn reinduces that same behavioral strategy. However, if a game lacks perfect recall, then not all mixed strategies can be induced from behavioral strategies.

5. Applications. When a game arises in applications, one frequently begins his analysis by describing the game in extensive form. This gives a rather complete picture of the problem, including the detailed structure or rules of the game, the sequence of all possible moves, the state of each player's information, and the payoffs. Except for very elementary games, however, one does not normally pursue his investigation in the extensive form. This is because the mathematical problem of actually determining optimal strategies for this form in an efficient manner has not been resolved.

Instead, one reduces the game to its normal form by considering the full list of pure strategies for each player and the resulting payoffs when these strategies are employed. In theory it is a much simpler problem to compute optimal strategies in the normal form. As mentioned, in the case of the two-person zero-sum games, it is equivalent to solving a pair of dual linear programs.

Good algorithms also exist in the case of two-person, general-sum games. For larger numbers of players, it is usually difficult to determine optimal solutions, and this is still an active area of research for games in normal form. Furthermore, in practice one usually gets an enormous increase in the number of parameters necessary to describe a mixed strategy in the resulting normal form compared to the number of variables necessary to determine a behavioral strategy in the original game tree. For example, H. W. Kuhn has described a trivial poker game in which the simplex of mixed strategies has dimension 8191 while the "cube" of behavioral strategies has dimension 13. This astronomical increase in the number of variables to be determined actually occurs in some important real-world problems and often forces the analyst to abandon the game theoretic approach.

In addition, even if the reduction to normal form results in a problem which is computationally feasible, there is still a major concern about whether this approach is philosophically sound. It is true that for games with perfect recall an optimal mixed strategy induces a behavioral strategy which will achieve the same expected payoff. It is questionable, at least in some people's minds, whether one would or should always play throughout the game according to this latter strategy.

Some of the many applications of the games in extensive form have been to areas of pure mathematics, including topics in logic as well as in the very foundations of mathematics. Certain problems in the area of combinatorial games relate to questions on computational complexity. Some of the developments in this direction are indicated in the recent book by J. H. Conway [3]. It has been shown that certain game trees of infinite length and perfect information need not have a pure strategy optimal solution if one assumes the axiom of choice. This has given rise in turn to alternate

assumptions and to questions about the relationship between such fundamental assumptions, which are most basic to much modern mathematics. A paper by Mycielski [13] gives an indication of some results and problems in this latter direction.

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